Solomon I. Khmelnik

The existence and the search method for global solutions of Navier-Stokes equation

Annotation

We formulate and prove the variational extremum principle for viscous incompressible and compressible fluid, from which principle follows that the Naviet-Stokes equations represent the extremum conditions of a certain functional. We describe the method of seeking solution for these equations, which consists in moving along the gradient to this functional extremum. We formulate the conditions of reaching this extremum, which are at the same time necessary and sufficient conditions of this functional global extremum existence.

Then we consider the so-called closed systems. We prove that for them the necessary and sufficient conditions of global extremum for the named functional always exist. Accordingly, the search for global extremum is always successful, and so the unique solution of Naviet-Stokes is found.

We contend that the systems described by Naviet-Stokes equations with determined boundary solutions (pressure or speed) on all the boundaries, are closed systems. We show that such type of systems include systems bounded by impenetworkrable walls, by free space under a known pressure, by movable walls under known pressure, by the so-called generating surfaces, through which the fluid flow passes with a known speed.

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Introduction

In his previous works [6-8] the author presented the full action extremum principle, allowing to construct the functional for various physical systems, and, which is most important, for dissipative systems. In [4, 5, 9] this principle is described applied to the hydrodynamics of incompressible fluid. There one may find multiple examples of this principle use for the solution of specific problems. In this paper the author is using a more strict extension of this principle for the powers [6] and considers also the hydrodynamics of incompressible fluid.

Here we are discussing the Navier-Stokes equations for viscous incompressible fluid. We show that these equations are the conditions of a certain functional's extremum. A solution method for these equation is described - it consists of moving by the gradient in the direction of this functional's extremum. The conditions of reaching this extremum are formulated – they are simultaneously necessary and sufficient conditions of the existence of this functional's global extremum.

Then we separate the so-called closed systems. For them it is proved that the necessary and sufficient conditions of the existence of this functional's global extremum are always valid Thereafter, the method of searching for global extremum always gives a positive result, and hence the sole solution of the Navier-Stokes equations is found.

It is stated that the systems described by Navier-Stokes equations, having definite boundary conditions (of pressure or speed) on all the boundaries, are closed systems. It is shown that such systems include the systems bounded by

- impenetrable walls ,
- free surfaces that are under known pressure,
- movable walls that are under known pressure,
- so-called generating surfaces through which the flows passes with known speed.

In this way we have shown that the Navier-Stokes equations have only one solution.

1. Viscous incompressible fluid

1.1. Hydrodynamic equations for viscous incompressible fluid

The hydrodynamic equations for viscous incompressible liquid are as follows [2]:

$$\operatorname{div}(v)=0, \tag{1}$$

$$\rho\frac{\partial v}{\partial t}+\nabla p-\mu\Delta v+\rho(v\cdot\nabla)v-\rho F=0, \tag{2}$$

where

$\rho=\mathrm{const}$ is constant density,

μ - coefficient of internal friction,

p - unknown pressure,

$v=\lfloor v_x, v_y, v_z\rfloor$ - unknown speed, vector,

$F=\lfloor F_x, F_y, F_z\rfloor$ - known mass force, vector,

x, y, z, t - space coordinates and time.

1.2. The power balance

Umov [1] discussed for the liquids the condition of balance for specific (by volume) powers in a liquid flow. For a non-viscous and incompressible liquid this condition is of the form (see (56) in [1])

$$P_1(v)+P_5(v)+P_4(p,v)=0, \tag{3}$$

and for viscous and incompressible liquid - another form (see (80) in [1])

$$P_1(v)+P_5(v)+P_2(p,v)=0, \tag{4}$$

where

$$P_1=\frac{\rho}{2}\frac{\partial W^2}{\partial t}, \tag{5}$$

$$P_2 = \left\{ \begin{array}{l} v_x \left(\frac{dp_{xx}}{dx} + \frac{dp_{xy}}{dy} + \frac{dp_{xz}}{dz} \right) + \\ v_y \left(\frac{dp_{xy}}{dx} + \frac{dp_{yy}}{dy} + \frac{dp_{yz}}{dz} \right) + \\ v_z \left(\frac{dp_{xz}}{dx} + \frac{dp_{yz}}{dy} + \frac{dp_{zz}}{dz} \right) \end{array} \right\} \tag{6}$$

$$P_4 = v \cdot \nabla p, \tag{7}$$

$$P_5 = \frac{1}{2}\rho \left(v_x \frac{dW^2}{dx} + v_y \frac{dW^2}{dy} + v_z \frac{dW^2}{dz} \right), \tag{8}$$

$$W^2 = \left(v_x^2 + v_y^2 + v_z^2 \right) \tag{9}$$

p_{xy} and so on – tensions (see [2]).

Here P_1 is the power of energy variation, P_4 is the power of work of pressure variation, P_5 - the power of variation of energy variation for direction change, and the value

$$P_7(p,v) = P_5(v) + P_4(p,v) \tag{10}$$

is, as it was shown by Umov, the variation of energy flow power through a given liquid volume – see (56) и (58) в [1]. In [2] it was shown, that for incompressible liquid the following equality is valid

$$\left| \begin{array}{l} \left(\frac{dp_{xx}}{dx} + \frac{dp_{xy}}{dy} + \frac{dp_{xz}}{dz} \right) \\ \left(\frac{dp_{xy}}{dx} + \frac{dp_{yy}}{dy} + \frac{dp_{yz}}{dz} \right) \\ \left(\frac{dp_{xz}}{dx} + \frac{dp_{yz}}{dy} + \frac{dp_{zz}}{dz} \right) \end{array} \right| = \nabla p - \mu \cdot \Delta v \cdot \tag{11}$$

From this it follows that

$$P_2 = v\left(\nabla p - \mu \cdot \Delta v\right) \tag{12}$$

or, subject to (6),

$$P_2 = P_4 - P_3 \tag{13}$$

where

$$P_3 = \mu \cdot v \cdot \Delta v \tag{14}$$

- power of change of energy loss for internal friction during the motion. Therefore, we rewrite (4) in the form

$$P_1(v) + P_5(v) + P_4(p, v) - P_3(v) = 0, \tag{15}$$

We shall supplement the condition (15) by mass forces power

$$P_6 = \rho F v. \tag{16}$$

Then for every viscous incompressible liquid this balance condition is of the form

$$P_1(v) + P_5(v) + P_4(p, v) - P_3(v) - P_6(v) = 0. \tag{17}$$

Taking into condition(1) and formula (p1a) let us rewrite (7) in the form

$$P_4 = \operatorname{div}(v \cdot p), \tag{18}$$

Taking into account (p9a), condition(1) and formula (p1a) let us rewrite (8) in the form

$$P_5 = \operatorname{div}\left(v \cdot W^2\right). \tag{19}$$

From (18, 19) and Ostrogradsky formula (p28) we find:

$$\iiint\limits_V P_4 dV = \iiint\limits_V \operatorname{div}(v \cdot p) dV = \iint\limits_S p \cdot v_n \cdot dS, \tag{20}$$

$$\iiint\limits_V P_5 dV = \iiint\limits_V \operatorname{div}\left(v \cdot W^2\right) dV = \iint\limits_S W^2 \cdot v_n \cdot dS \tag{20a}$$

or, subject to (p15),

$$\iiint\limits_V P_5 dV = \iiint\limits_V (v \cdot G(v)) dV = \iint\limits_S W^2 \cdot v_n \cdot dS. \tag{21}$$

Returning again to the definitions of powers (7, 8), we will get

$$\iiint\limits_V (v \cdot \nabla p) dV = \iint\limits_S p_S \cdot v_n \cdot dS, \tag{21a}$$

$$\iiint\limits_V \left(v \cdot \nabla\left(W^2\right)\right) dV = \iint\limits_S W^2 \cdot v_n \cdot dS \tag{21в}$$

or

$$\iiint\limits_V (v \cdot G(v)) dV = \iint\limits_S W^2 \cdot v_n \cdot dS. \tag{21c}$$

1.3. Energian-2 and quasiextremal

For further discussion we shall assemble the unknown functions into a vector

$$q = \lfloor p, v \rfloor = \lfloor p, v_x, v_y, v_z \rfloor. \qquad (22)$$

This vector and all its components are functions of (x, y, z, t). We are considering a liquid flow in volume V. The full action-2 [6] in hydrodynamics takes a form

$$\Phi = \int_0^T \left\{ \oint_V \Re(q(x, y, z, t) dV \right\} dt, \qquad (23)$$

Having in mind (17) the definition of energian-2 in [6], let us write the energian-2 in the following form:

$$\Re(q) = P_1(v) - \frac{1}{2} P_3(v) + P_4(q) + P_5(v) - P_6(v). \qquad (24)$$

Below in Supplement 1 will be shown – see (p4, p13, p15):

$$P_1 = \rho \cdot v \frac{dv}{dt}, \qquad (25)$$

$$P_5 = \rho \cdot v \cdot G(v), \qquad (26)$$

where

$$G(v) = (v \cdot \nabla) v. \qquad (27)$$

Taking this into account let us rewrite the energian-2 (24) in a detailed form

$$\Re(q) = \rho \cdot v \frac{dv}{dt} - \frac{1}{2} \mu \cdot v \cdot \Delta v + \operatorname{div}(v \cdot p) + \rho \cdot v \cdot G(v) - \rho F v. \qquad (28)$$

Further we shall denote the derivative computed according to Ostrogradsky formula (p23), by the symbol $\frac{\partial_o}{\partial v}$, as distinct from ordinary derivative $\frac{\partial}{\partial v}$. Taking this into account (p16), we get

$$\left. \begin{array}{l} \dfrac{\partial}{\partial v}\left(P_1\left(v, \dfrac{dv}{dt} \right) \right) = \rho \dfrac{dv}{dt}; \quad \dfrac{\partial_o}{\partial v}(P_3(v)) = \mu \cdot \Delta v; \\ \dfrac{\partial}{\partial q}(P_4(q)) = \begin{vmatrix} \operatorname{div}(v) \\ \nabla(p) \end{vmatrix}; \quad \dfrac{\partial}{\partial v}(P_5(v, G(v))) = \rho (v \cdot \nabla) v; \\ \dfrac{\partial_o}{\partial v}(P_6(v)) = \rho F. \end{array} \right\} \qquad (29)$$

In accordance with [6] we write the quasiextremal in the following form:

$$\left[\begin{array}{l}\frac{\partial}{\partial v}\left(P_1\left(v,\frac{dv}{dt}\right)\right)+\frac{1}{2}\frac{\partial_o}{\partial v}\left(P_3(v)\right)+\frac{\partial}{\partial q}\left(P_4(q)\right)\\ +\frac{\partial}{\partial v}\left(P_5(v,G(v))\right)-\frac{\partial_o}{\partial v}\left(P_6(v)\right)\end{array}\right]=0. \tag{30}$$

From (29) it follows that the quasiextremal (30) after differentiation coincides with equations (1, 2).

1.4. The split energian-2

Let us consider the split functions (22) in the form

$$q'=[p',v']=\lfloor p',v'_x,v'_y,v'_z\rfloor, \tag{31}$$

$$q''=[p'',v'']=\lfloor p'',v''_x,v''_y,v''_z\rfloor. \tag{32}$$

Let us present the split energian-2 taking into account the formula (p15) in the form

$$\Re_2(q',q'')=\left\{\begin{array}{l}\rho\cdot\left(v'\frac{dv''}{dt}-v''\frac{dv'}{dt}\right)-\mu\cdot\left(v'\Delta v'-v''\Delta v''\right)\\ +2\left(\mathrm{div}\left(v'\cdot p''\right)-\mathrm{div}\left(v''\cdot p'\right)\right)+\\ \rho\cdot\left(v'G(v'')-v''G(v')\right)-\rho\cdot F\left(v'-v''\right)\end{array}\right\}. \tag{33}$$

Let us associate with the functional (23) functional of <u>split full action-2</u>

$$\Phi_2=\int_0^T\left\{\oint_V\Re_2(q',q'')dV\right\}dt, \tag{34}$$

With the aid of Ostrogradsky formula (p23) we may find the variations of functional (34) with respect to functions q'. In this we shall take into account the formulas (p21), obtained in the Supplement 1. Then we have:

$$\frac{\partial_o\Re_2}{\partial p'}=b_{p'}, \tag{35}$$

$$\frac{\partial_o\Re_2}{\partial v'}=b_{v'}, \tag{36}$$

$$b_{p'}=2\mathrm{div}(v''), \tag{37}$$

$$b_{v'} = \left\{ \begin{array}{l} 2\rho \cdot \frac{dv''}{dt} - 2\mu \cdot \Delta v' + 2\nabla(p'') \\ + 2\rho \cdot \left[G\left(v'', \frac{\partial v''}{\partial X} \right) + G\left(v', \frac{\partial v''}{\partial X} \right) \right] - \rho \cdot F \end{array} \right\}. \tag{38}$$

So, the vector

$$b' = \lfloor b_{p'}, b_{v'} \rfloor \tag{39}$$

is a variation of functional (34), and the condition

$$b' = \lfloor b_{p'},\ b_{v'} \rfloor = 0 \tag{40}$$

is the necessary condition for the existence of the extremal line. Similarly,

$$b'' = \lfloor b_{p''},\ b_{v''} \rfloor = 0 \tag{41}$$

The equations (40, 41) are necessary condition for the existence of a saddle line. By symmetry of these equations we conclude that the optimal functions q_0' and q_0'', satisfying these equations, satisfy also the condition

$$q_0' = q_0''. \tag{42}$$

Subtracting in couples the equations (40, 41) taking into consideration (37, 38), we get

$$2\mathrm{div}(v' + v'') = 0, \tag{43}$$

$$\left\{ \begin{array}{l} 2\rho \cdot \frac{d(v' + v'')}{dt} - 2\mu \cdot \Delta(v' + v'') + 2\nabla(p' + p'') - 2\rho \cdot F \\ + 2\rho \cdot \left[G\left(v'', \frac{\partial v''}{\partial X} \right) + G\left(v', \frac{\partial v''}{\partial X} \right) + G\left(v', \frac{\partial v'}{\partial X} \right) + G\left(v'', \frac{\partial v'}{\partial X} \right) \right] \end{array} \right\} = 0. \tag{44}$$

For $v' = v''$ according to (p12), we have

$$\left[G(v'') + G\left(v', \frac{\partial v''}{\partial X} \right) + G(v') + G\left(v'', \frac{\partial v'}{\partial X} \right) \right] = G(v' + v''). \tag{45}$$

Taking into account (27, 45) and reducing (43, 44) by 2, получаем we get the equations (1, 2), where

$$q = q_o' + q_o'' \tag{46}$$

- see (22, 31, 32), i.e. the equations of extremal line are Naviet-Stokes equations.

1.5. About sufficient conditions of extremum

Let us rewrite the functional (34) in the form

$$\Phi_2 = \int_0^T \left\{ \oint_z \left\{ \oint_y \left\{ \oint_x \Re_2(q', q'') dx \right\} dy \right\} dz \right\} dt, \tag{47}$$

где векторы q', q'' определенны по (31, 32), $X = (x, y, z, t)$ – вектор независимых переменных. Далее будем варьировать только функции $q'(X) = [p'(X), v'(X)]$.

Vector b, defined by (39), is a variation of functional Φ_2 by the function q' and depends on function q', i.e. $b = b(q')$. Here the function q'' here is fixed.

Let S be an extremal, and subsequently, the gradient in it is $b_S = 0$. To find out which type of extremum we have, let us look at the sign of functional's increment

$$\delta\Phi_2 = \Phi_2(S) - \Phi_2(C), \tag{48}$$

where C is the line of comparison, where $b = b_c \neq 0$. Let the values vector q' on lines S и C differ by

$$q'_C - q'_S = q' - q'_S = \delta q' = a \cdot b, \tag{49}$$

where b is the variation on the line C, a – a known number. Thus,

$$q' = q'_S + a \cdot b = \begin{vmatrix} p'_S \\ v'_S \end{vmatrix} + a \begin{vmatrix} b_p \\ b_v \end{vmatrix}. \tag{50}$$

where b_p, b_v are determined by (35, 36) accordingly, and do not depend on q'.

If

$$\delta\Phi_2 = a \cdot A, \tag{51}$$

where A has a constant sign in the vicinity of extremal $b_S = 0$, then this extremal is sufficient condition of extremum. If, furthermore, A is of constant sign in all definitional domain of the function q', then this extremal determines a global extremum.

From (48) we find

$$\delta\Re_2 = \Re_2(S) - \Re_2(C) = \Re_2(q'_S) - \Re_2(q'), \tag{52}$$

or, taking into account (33, 50),

$$\delta\Re_2 = \begin{Bmatrix} -\rho\cdot\left(\left(v'_s+ab_v\right)\frac{dv''}{dt}-v''\frac{d\left(v'_s+ab_v\right)}{dt}\right) \\ -\mu\cdot\left(\left(v'_s+ab_v\right)\Delta\left(v'_s+ab_v\right)-v''\Delta\left(v''\right)\right) \\ +2\left(\left(v'_s+ab_v\right)\cdot\nabla\left(p''\right)-v''\cdot\nabla\left(p'_s+ab_p\right)\right) \\ +2\rho\cdot\left(\left(v'_s+ab_v\right)G\left(v''\right)-v''G\left(v'_s+ab_v\right)\right) \\ -\rho\cdot F\left(\left(v'_s+ab_v\right)-v''\right) \end{Bmatrix} \tag{53}$$

Taking into account (p20), we get:

$$G(v'_s+ab_v)=G(v'_s)+a\left[G_1\left(v'_s,b_v\right)+G_2\left(v'_s,b_v\right)\right]+a^2G(b_v). \tag{54}$$

Here (53) is transformed into

$$\delta\Re_2=\Re_{20}+\Re_{21}a+\Re_{22}a^2, \tag{55}$$

where $\Re_{20}$, $\Re_{21}$, $\Re_{22}$ are functions not dependent on *a*, of the form

$$\Re_{20}=\begin{Bmatrix} \rho\cdot\left(v'_s\frac{dv''}{dt}-v''\frac{d\left(v'_s\right)}{dt}\right) \\ -\mu\cdot\left(v'_s\Delta\left(v'_s\right)-v''\Delta\left(v''\right)\right)+2\left(v'_s\nabla\left(p''\right)-v''\cdot\nabla\left(p'_s\right)\right) \\ +2\rho\cdot\left(v'_sG(v'')-v''G\left(v'_s\right)\right)-\rho\cdot F\left(v'_s-v''\right) \end{Bmatrix}, \tag{56}$$

$$\Re_{21}=\begin{Bmatrix} \rho\cdot\left(b_v\frac{dv''}{dt}-v''\frac{db_v}{dt}\right)-\mu\cdot\left(b_v\Delta v'_s+v'_s\Delta\left(b_v\right)\right) \\ +2\left(b_v\cdot\nabla\left(p''\right)-v''\cdot\nabla\left(b_p\right)\right)+ \\ 2\rho\left(b_vG(v'')-v''\left(G_1\left(v'_s,b_v\right)+G_2\left(v'_s,b_v\right)\right)\right)-\rho\cdot F\cdot b_v \end{Bmatrix}, \tag{57}$$

$$\Re_{22}=-\mu b_v\Delta(b_v)-2\rho v''G(b_v). \tag{58}$$

Now we must find

$$\frac{\partial^2\left(\delta\Re_2\right)}{\partial a^2}=\Re_{22} \tag{59}$$

This function depends on q'. To prove that the necessary condition (40) is also a sufficient condition of global extremum of the functional (47) with respect to function q', we must prove that the integral

$$\frac{\partial^2 \Phi_2}{\partial a^2} = \int_0^T \left\{ \oint_V \partial \Re_2(q', q'') dV \right\} dt \qquad (60)$$

or, which is the same, the integral

$$\frac{\partial^2 \Phi_2}{\partial a^2} = \int_0^T \left\{ \oint_V \Re_{22} dV \right\} dt \qquad (61)$$

is of constant sign. Similarly, to prove that the necessary condition (41) is also a sufficient condition of a global extremum of the functional (47) with respect to function q'', we have to prove that the integral similar to (60) is also of the same sign.

Specifying the concepts, we will say that the Navier-Stokes equations have a global solution, if for them there exists a unique non-zero solution in a given domain of the fluid existence.

In the above-cited integrals the energy flow through the domain's boundaries was not taken into account. Hence the above-stated may be formulated as the following lemma

Lemma 1. The Navier-Stokes equations for incompressible fluid have a global solution in an unlimited domain, if the integral (61, 58) has constant sign for any speed of the flow.

2. Boundary conditions

The boundary conditions determine the power flow through the boundaries, and, generally speaking, they may alter the power balance equation. Let us view some specific cases of boundaries.

2.1. Absolutely hard and impenetrable walls

If the speed has a component normal to the wall, then the wall gets energy from the fluid, and fully returns it to the fluid. (changing the speed direction). The tangential component of speed is equal to zero (adhesion effect). Therefore such walls do not change the system's energy. However, the energy reflected from walls creates an internal energy flow, circulating between the walls. So in this case all the above-stated formulas remain unchanged, but the conditions on the walls (impenetrability, adhesion) should not be formulated explicitly – they appear as a result of solving the problem with integrating in a domain bounded by walls. Then the second lemma is valid:

Lemma 2. The Navier-Stokes equations for incompressible fluid have a global solution in a domain bonded by absolutely hard and impenetrable walls, if the integral (61, 58) is of the same sign for any flow speed.

2.2. Systems with a determined external pressure

In the presence of external pressure the power balance condition (17) is supplemented by one more component – the power of pressure forces work

$$P_8 = p_s \cdot v_n, \tag{62}$$

where

p_s - external pressure,

S - surfaces where the pressure determined,

v_n - normal component of flow incoming into above surface,

In this case the full action-2 is presented as follows:

$$\Phi = \int_0^T \left\{ \oint_V \Re(q(x,y,z,t)dV + \oint_S P_8(q(x,y,z,t)dV \right\} dt. \tag{63}$$

For convenience sake let us consider the functions Q, determined on the domain of the flow existence and taking zero value in all the points of this domain, except the points belonging to the surface S. Then the restraint (63) may be written in the form

$$\Phi = \int_0^T \left\{ \oint_V \hat{\Re}(q(x,y,z,t)dV \right\} dt, \tag{64}$$

where energian-2

$$\hat{\Re}(q) = \Re(q) + Q \cdot P_8(v_n). \tag{65}$$

One may note that here the last component is identical to the power of body forces – in the sense that both of them depend only on the speed. So all the previous formulas may be extended on this case also, by performing substitution in them.

$$F \Rightarrow F + Q \cdot p_s / \rho. \tag{66}$$

Therefore the following lemma is true:

Lemma 3. The Navier-Stokes equations for incompressible fluid have a global solution in a domain bounded by surfaces with a certain pressures, if the integral (61, 58) has constant sign for any flow speed.

Such surface may be a free surface or a surface where the pressure is determined by the problem's conditions (for example, by a given pressure in the pipe section).

Note also that the pressure p_S may be included in the full action functional formally, without bringing in physical considerations. Indeed, in the presence of external pressure there appears a new constraint - (21a). In [3] it is shown that such problem of a search for a certain functional with integral constraints (certain integrals of fixed values) is equivalent to the search for the extremum of the of the sum of our functional and integral constraint. More precisely, in our case we must seek for the extremum of the following functional:

$$\Phi = \int_0^T \left\{ \oint_V \hat{\Re}(q(x,y,z,t)) dV \right\} dt, \tag{67}$$

$$\hat{\Re}(q(x,y,z,t) = \left\{ \begin{array}{l} \Re(q(x,y,z,t)) + \\ \lambda \cdot \left(- v \cdot \nabla p + Q \cdot p_S \cdot v_n\right) \end{array} \right\}, \tag{68}$$

where λ – an unknown scalar multiplier. It is determined or known initial conditions [3]. For $\lambda = 1$ after collecting similar terms the energian-2 (68) again assumes the form (65), which was to be proved.

2.3. Systems with generating surfaces

There is a conception often used in hydrodynamics of a certain surface through which a flow enters into a given fluid volume with a certain constant speed, i.e., NOT dependent on the processes going on in this volume. The energy entering into this volume with this flow, evidently will be proportional to squared speed module and is constant. We shall call such surface a generating surface (note that this is to some extent similar to a source of stabilized direct current whose magnitude does not depend on the electric circuit resistance).

If there is a generating surface, the power balance condition (17) is supplemented by another component – the power of flow with constant squared speed module.

$$P_9 = W_s^2 \cdot v_n, \tag{69}$$

где

W_S - squared module of input flow speed,

S - surfaces where the pressure determined,

v_n - normal component of flow incoming into above surface,

One may notice a formal analogy between W_S and p_S. So here we also may consider the functional (64), where the energian-2 is

$$\hat{\Re}(q) = \Re(q) + Q \cdot P_9(v_n), \tag{70}$$

and then perform the substitution

$$F \Rightarrow F + Q \cdot W_S^2 / \rho. \tag{71}$$

Consequently, the following lemma is true:

Lemma 4. The Navier-Stokes equations for incompressible fluid have a global solution in a domain bounded by generating surface with a certain pressure , if the integral (61, 58) has constant sign for any flow speed.

Note also that W_S the pressure p_S may be included in the full action-2 functional formally, without bringing in physical considerations.(similar with pressure p_S). Indeed, in the presence of external pressure there appears a new constraint - (21c). Including this integral constraint into the problem of the search for functional's extremum, we again get energian-2 (70).

2.4. Closed systems

We will call the system closed if it is bounded by

- absolutely hard and impenetrable walls,
- surfaces with certain external pressure,,
- generating surfaces, or
- not bounded by anything.

In the last case the system will be called absolutely closed. Such case is possible. For example, local body forces in a bondless ocean create such a system, and we shall discuss this case later. There is a possible case when the system is bounded by walls, but there is no energy exchange between fluid and walls. An example – a flow in endless pipe under the action of axis body forces Such example will also be considered below.

In consequence of Lemmas 1-4, the following theorem is true:

Theorem 1. The Navier-Stokes equations for incompressible fluid have a global solution in a given domain, if

- the domain of fluid existence is closed,

- the integral (61, 58) has constant sign for any flow speed.

The free surface, which is under certain pressure, may also be the boundary of a closed system. But the boundaries of this system are changeable, and the integration must be performed within the fluid volume. It is well known that the fluid flow through a certain surface S is determined as

$$w_S = \oiint_S \rho \cdot \mathrm{div}(v) \cdot d\Theta . \tag{72}$$

Thus, the boundary conditions in the form of free surface are fully considered, by the fact that the integration must be performed within the changeable boundaries of the free surface.

We have indicated above, that the power of energy flow change is determined by (10). In a closed system this power is equal to zero. Therefore for such system the energian-2 (24) or (28) turns into energian-2 (accordingly)

$$\Re(q) = P_1(v) + P_3(v) - P_6(v), \tag{73}$$

$$\Re(q) = \rho \cdot v \frac{dv}{dt} + \mu \cdot v \cdot \Delta v - \rho F v . \tag{74}$$

For such systems the Navier-Stokes equations take the form (1) and

$$\rho \frac{\partial v}{\partial t} - \mu \Delta v - \rho F = 0, \tag{75}$$

Some examples of such system will be cited below.

3. Modified Navier-Stokes equations

From (p19a) we find that

$$(v \cdot \nabla) \cdot v = \Delta\left(W^2\right)/2 . \tag{76}$$

Substituting (76) in (2), we get

$$(v \cdot \nabla) \cdot v = \Delta\left(W^2\right)/2 . \tag{77}$$

Let us consider the value

$$D = \left(p + \frac{\rho}{2} W^2 \right), \tag{78}$$

which we shall call quasipressure. Then (77) will take the form

$$\rho \frac{\partial v}{\partial t} - \mu \cdot \Delta v + \nabla D - \rho \cdot F = 0 . \tag{79}$$

The equations system (1, 79) will be called modified Navier-Stokes equations. The solution of this system are functions v, D, and the pressure may be determined from (9, 78). It is easy to see that the equation (79) is much simpler than (2).

The above said may be formulated as the following lemma.

Lemma 5. If a given domain of incompressible fluid is described by Navier-Stokes equations, then it is also described by modified Navier-Stokes equations, and their solutions are similar.

Physics aside, we may note that from mathematical point of view the equation (79) is a particular case of equation (2), and so all the previous reasoning may be repeated for modified Navier-Stokes equations. Let us do it.

The functional of split full action-2 (34) contains modified split energian-2

$$\Re_2(q',q'') = \left\{ \begin{array}{l} -\rho \cdot \left(v' \dfrac{dv''}{dt} - v'' \dfrac{dv'}{dt} \right) - \mu \cdot \left(v' \Delta v' - v'' \Delta v'' \right) \\ + \left(\operatorname{div}(v' \cdot D'') - \operatorname{div}(v'' \cdot D') \right) - \rho \cdot F (v' - v'') \end{array} \right\}. \tag{80}$$

- see (33). Gradient of this functional with respect to function q' is (37) and

$$b_{v'} = \left\{ 2\rho \cdot \frac{dv''}{dt} - 2\mu \cdot \Delta v' + 2\nabla(D'') - \rho \cdot F \right\}. \tag{81}$$

- see (38). The components of equation (55) take the form

$$\Re_{21} = \left\{ \begin{array}{l} -\rho \cdot \left(b_v \dfrac{dv''}{dt} - v'' \dfrac{db_v}{dt} \right) - \mu \cdot \left(b_v \Delta v_s' + v_s' \Delta(b_v) \right) \\ + 2\left(b_v \cdot \nabla(D'') - v'' \cdot \nabla(b_p) \right) - \rho \cdot F \cdot b_v \end{array} \right\}, \tag{82}$$

$$\Re_{22} = -\mu b_v \Delta(b_v). \tag{83}$$

Thus, for modified Navier-Stokes equations by analogy with Theorem 1 we may formulate the following theorem

Theorem 2. Modified Navier-Stokes equations for incompressible fluid have a global solution in the given domain, if

- the fluid domain of existence is a closed system
- the integral (61, 83) has the same sign for any fluid flow speed.

Lemma 6. Integral (61, 83) always has positive value.

Proof. Consider the integral

$$J = \mu \int_0^T \left\{ \oint_V v \cdot \Delta(v) dV \right\} dt \tag{84}$$

This integral expresses the thermal energy, evolved by the liquid due to internal friction. This energy is positive not depending on what function connects the vector of speeds with the coordinates. A stricter proof of this statement is given in Supplement 3. Hence, integral (84) is positive for any speed. Substituting in (84) $v = b_v$, we shall get integral (61, 83), which is always positive, as was to be proved.

From Lemmas 5, 6 and Theorem 2 there follows a following.

Theorem 3. The equations of Navier-Stokes for incompressible fluid always have a solution in a closed domain.

The solution of equation (1, 79) permits to find the speeds. Calculation of pressures **inside** the closed domain with known speeds is performed with the aid of equation (78) or

$$\nabla p + \rho(v \cdot \nabla)v = 0. \tag{85}$$

4. Conclusions

1. Among the computed volumes of fluid flow the closed volumes of fluid flow may be marked, which do not exchange flow with adjacent volumes – the so-called closed systems.

2. The closed systems are bounded by:
   - Impenetworkrable walls,
   - Surfaces, located under the known pressure,
   - Movable walls being under a known pressure,
   - So-called generating surfaces through which the flow passes with a known speed.

3. It may be contended that the systems described by Naviet-Stokes equations, and having certain boundary conditions (pressures or speeds) on all boundaries, are closed systems.

4. For closed systems the global solution of modified Navier-Stokes equations always exists.

5. The solution of Navier-Stokes equations may always be found from the solution of modified Navier-Stokes equations. Therefore, for closed systems there always exists a global solution of modified Navier-Stokes equations.

5. Computational Algorithm

The method of solution for hydrodynamics equations with a known functional, having a global saddle point, is based on the following outlines [7, 8]. For the given functional from two functions q_1, q_2 two more secondary functionals are formed from those functions

q_1, q_2. Each of these functionals has its own global saddle line. Seeking for the extremum of the main functional is substituted by seeking for extremums of two secondary functionals, and we are moving simultaneously along the gradients of these functionals. In general operational calculus should be used for this purpose. However, in some particular cases the algorithm may be considerably simplified.

Another complication is caused by the fact that in the computations we have to integrate over all the flow area. But the area may be infinite, and full integration is impossible. Nevertheless, the solution is possible also for an infinite area, if the flow speed is damping.

The solution method consists in moving along the gradient towards saddle point of the functional generated from the power balance equation. The obtained solutions:

a. may be interpreted as experimentally found physical effects (for instance, the walls impermeability, "sticking" of fluid to the walls, absence of energy flow through a closed system),
b. coincide with solutions obtained earlier with the aid of other methods (for instance, the solution of Poiseille problem),
c. may иe seen as generalization of known solutions (for instance, a generalization of Poiseille problem solution for pipes with arbitrary form of section and/or with arbitrary form of axis line),
d. belong to unsolved (as far as the author knows) problems (for instance, problems with body as the functions of speed, coordinates and time) .

Here we shall discuss only these particular cases.
Various examples are given in [4, 5].

6. Stationary Problems

Note that in stationary mode the equations (2.1, 2.2) assumes the form

$$\begin{cases} \operatorname{div}(v) = 0, \\ \nabla p - \mu\Delta v + \rho(v \cdot \nabla)v - \rho F = 0 \end{cases} \tag{1}$$

The modified equations (1, 79) in stationary mode take the form:

$$\begin{cases} \operatorname{div}(v) = 0, \\ -\mu \cdot \Delta v + \nabla D - \rho \cdot F = 0. \end{cases} \tag{2}$$

In Appendix 6 we considered the discrete version of modified Navier-Stokes equations for stationary systems (2). It was shown that for stationary closed systems the solution of modified Navier-Stokes equations is reduced to a search for quadratic functional minimum (and not a saddle points, as in general case). After solving these equations the pressure is calculated by the equation (2.78), i.e.

$$p = D - \frac{\rho}{2} W^2 . \tag{3}$$

or

$$\nabla p = \nabla D - \rho (v \cdot \nabla) v = 0 \tag{4}$$

The equation (75) for absolutely closed systems in stationary mode takes the form

$$-\mu \Delta v - \rho F = 0 . \tag{5}$$

The solution of equation (2) has been discussed in detail in Supplement 4. After solving it the pressures are calculated by the equation (4).

7. Dynamic Problems

7.1. Absolutely closed systems

Let us consider the equation (2.75) for absolutely closed systems and rewrite is as

$$\frac{\partial v}{\partial t} - \eta \Delta v - F = 0 \tag{1}$$

where

$$\eta = \frac{\mu}{\rho} . \tag{2}$$

Assuming that time is a discrete variable with step dt, we shall rewrite (1) as

$$\frac{v_n - v_{n-1}}{dt} - \eta \Delta v_n - F_n = 0 , \tag{3}$$

where $n = 1,2,3,...$ – the number of a time point. Let us write (3) as

$$\frac{v_n}{dt} - \eta \cdot \Delta v_n - F_{n1} = 0 . \tag{4}$$

where

$$F_{n1} = \left(F_n + \frac{v_{n-1}}{dt} \right) . \tag{5}$$

For a known speed v_{n-1} the value v_n is determined by (4). Solving this equation is similar to solving a stationary problem. On the whole the algorithm of solving a dynamic problem for a closed system is as follows

<u>Algorithm 1</u>

1. v_{n-1} and F_n are known
2. Computing v_n by (4, 5).
3. Checking the deviation norm

$$\varepsilon = \frac{\partial v_n}{\partial t} - \frac{\partial v_{n-1}}{\partial t} \tag{6}$$

and, if it doesn't exceed a given value, the calculation is over. расчет заканчивается. Otherwise we assign

$$v_{n-1} \Leftarrow v_n \tag{7}$$

and go to p. 1.

Example 1. Let the body forces on a certain time point assume instantly a certain value – there is a jump of body forces. Then in the initial moment the speed $v_o = 0$, and on the first iteration we assign $v_{n-1} = 0$. Further we perform the computation according to Algorithm 1.

7.2. Closed systems with variable mass forces and external pressures

Consider the modified equation (1, 79) in the case when the mass forces are sinusoidal functions of time with circular frequency ω. In this case equations (1, 79) take the form of equations with complex variables:

$$\begin{cases} \operatorname{div}(v) = 0, \\ j \cdot \omega \cdot \rho \cdot v - \mu \cdot \Delta v + \nabla D - \rho \cdot F = 0, \end{cases} \tag{8}$$

where j - the imaginary unit.

In [4, 5] the discrete version of these equations is considered. There it is shown that their solution is reduced to the search of saddle point of a certain function of complex variables. After solving these equations the pressure is calculated by equation(4).

8. Compressible fluid

In this section we shall use this principle for the Navier-Stokes equations describing compressible fluid.

Navier-stokes equation for viscous compressible fluid are considered. It is shown that these equations are the conditions of a certain functional's extremum. The method of finding the solution of these equations is described. It consists of moving along the gradient towards the extremum of his functional. The conditions of reaching this extremum are formulated – they are simultaneously necessary and sufficient conditions of the existence of this functional's global extremum

8.1. The equations of hydrodynamics

In contrast with the equations for viscous incompressible fluid, the equations for viscous compressible fluid have the following form [2]:

$$\frac{\partial\rho}{\partial \mathrm{t}}+\mathrm{div}(\rho\cdot v)=0, \tag{4}$$

$$\rho\frac{\partial v}{\partial t}+\nabla p-\mu\cdot\Delta v+\rho\cdot G(v)-\rho\cdot F-\frac{\mu}{3}\Omega(v)=0, \tag{5}$$

where

$$\Omega(v)=\nabla(\nabla v). \tag{6}$$

В приложении функции (3) и (6) представлены в развернутом виде – см. (р14, р29, р30). Для сжимаемой жидкости плотность является известной функцией давления:

$$\rho=f(p). \tag{7}$$

Further the reasoning will be by analogy with the previous. In this case we have to consider also the power of energy loss variation in the course of expansion/compression due to the friction.

$$P_8(v)=\frac{\mu}{3}v\cdot\Omega(v). \tag{9}$$

We have also:

$$\frac{\partial}{\partial v}\left(P_8(v)\right)=\frac{\mu}{3}\Omega(v). \tag{10}$$

We may note that the function $\Omega(v)$ in the present context behaved in the same way as the function $\Delta(v)$. This allows to apply the proposed method also for compressible fluids.

8.2. Energian-2 and quasiextremal

By analogy with previous reasoning we shall write the formula for quasiextremal for compressible fluid in the following form:

$$\left\{\begin{array}{l} \frac{\partial}{\partial v}\left(\rho\cdot v\frac{dv}{dt}\right)-\frac{1}{2}\mu\cdot\frac{\partial_o}{\partial v}(v\cdot\Delta v)+\frac{\partial}{\partial q}\left(\frac{1}{\rho}\operatorname{div}(\rho\cdot p\cdot v)\right)+ \\ +\frac{\partial}{\partial v}(\rho\cdot v\cdot G(v))-\frac{\partial_o}{\partial v}(\rho\cdot F\cdot v)- \\ -\frac{\partial}{\partial p}\left(\frac{p}{\rho}\frac{\partial\rho}{\partial t}\right)-\frac{1}{2}\frac{\mu}{3}\cdot\frac{\partial_o}{\partial v}(v\cdot\Omega(v)). \end{array}\right\}=0\cdot \quad (11)$$

8.3. The split energian-2

By analogy with previous reasoning we shall write the formula for split energian-2 for compressible fluid in the following form:

$$\Re_2(q',q'')=\left\{\begin{array}{l} \rho\cdot\left(v'\frac{dv''}{dt}-v''\frac{dv'}{dt}\right)-\mu\cdot(v'\Delta v'-v''\Delta v'') \\ +\frac{2}{\rho}((\operatorname{div}(\rho\cdot v'\cdot p'')-\operatorname{div}(\rho\cdot v''\cdot p')))+ \\ \rho\cdot(v'G(v'')-v''G(v'))-\rho\cdot F(v'-v'')- \\ \frac{2}{\rho}\left(p'\frac{d\rho}{dt}-p''\frac{d\rho}{dt}\right)-\frac{\mu}{3}\cdot(v'\Omega(v')-v''\Omega(v'')) \end{array}\right\}. \quad (12)$$

With the aid of Ostrogradsky formula (p23) we may find the variations of functional of <u>spilt full action-2</u> with respect to functions q':

$$\frac{\partial_o\Re_2}{\partial p'}=b_{p'}, \quad (13)$$

$$\frac{\partial_o\Re_2}{\partial v'}=b_{v'}, \quad (14)$$

These variations are determined by varying the functions p' and v', whereas the functions ρ, p'', v'' do not change. Then we shall get:

$$1)\ \frac{\partial}{\partial v'}\left[\rho\cdot\left(v'\frac{dv''}{dt}-v''\frac{dv'}{dt}\right)\right]=2\rho\frac{dv''}{dt},$$

$$2)\ \frac{\partial}{\partial v'}\left[-\mu\cdot\left(v'\Delta v'-v''\Delta v''\right)\right]=-2\mu\cdot\Delta v',$$

$$3)\ \frac{\partial}{\partial v'}\left[\rho\left(v'G(v'')-v''G(v')\right)\right]=2\rho\cdot\left[G\left(v'',\frac{\partial v''}{\partial X}\right)+G\left(v',\frac{\partial v''}{\partial X}\right)\right],$$

$$4)\ \frac{\partial}{\partial v'}\left[-\rho\cdot F\left(v'-v''\right)\right]=-\rho\cdot F,$$

$$5)\ \frac{\partial}{\partial v'}\left[-\frac{\mu}{3}\cdot\left(v'\Omega(v')-v''\Omega(v'')\right)\right]=-\frac{2\mu}{3}\cdot\Omega(v'),$$

$$6)\ \frac{\partial}{\partial v'}\left[\frac{2}{\rho}\left(\mathrm{div}(\rho\cdot v'\cdot p'')-\mathrm{div}(\rho\cdot v''\cdot p')\right)\right]=2\mathrm{grad}(p''),$$

$$7)\ \frac{\partial}{\partial p'}\left[\frac{2}{\rho}\left(\mathrm{div}(\rho\cdot v'\cdot p'')-\mathrm{div}(\rho\cdot v''\cdot p')\right)\right]=-\frac{2}{\rho}\mathrm{div}(\rho\cdot v''),$$

$$8)\ \frac{\partial}{\partial p'}\left[-\frac{2}{\rho}\left(p'\frac{d\rho}{dt}-p''\frac{d\rho}{dt}\right)\right]=-\frac{2}{\rho}\frac{d\rho}{dt}. \qquad (15)$$

Remarks for these formulas:

1, 2, 3, 4) – the derivation is given below,

5) – is similar to formula 2),

6, 7) – the derivation is given in the Supplement 1 – see (p34, p35) accordingly

Then we have:

$$b_{p'}=-2\frac{d\rho}{dt}-2\mathrm{div}(\rho\cdot v''), \qquad (16)$$

$$b_{v'}=\left\{\begin{matrix}2\rho\cdot\frac{dv''}{dt}-2\mu\cdot\Delta(v')-\frac{2\mu}{3}\cdot\Omega(v')+2\nabla(p'')\\ +2\rho\cdot\left[G\left(v'',\frac{\partial v''}{\partial X}\right)+G\left(v',\frac{\partial v''}{\partial X}\right)\right]-\rho\cdot F\end{matrix}\right\}. \qquad (17)$$

As was shown above, the condition

$$b'=\left[b_{p'},\ b_{v'}\right]=0 \qquad (18)$$

and the similar condition

$$b'' = \lfloor b_{p''},\ b_{v''} \rfloor = 0 \qquad (19)$$

Are necessary conditions for the existence of a saddle line. From the symmetry of these equations it follows that the optimal functions q'_0 and q''_0, satisfying the equations (18, 19), must satisfy also the condition

$$q'_0 = q''_0. \qquad (20)$$

Subtracting in pairs the equations (18, 19) taking into account (16, 17), we get

$$-2\frac{d\rho}{dt} - 2\operatorname{div}(v' + v'') = 0, \qquad (21)$$

$$\left\{\begin{array}{l} +2\rho\cdot\dfrac{d(v'+v'')}{dt} - 2\mu\cdot\Delta(v'+v'') - \dfrac{2\mu}{3}\cdot\Omega(v'+v'') + \\ +2\nabla(p'+p'') - 2\rho\cdot F + 2\rho\cdot\left[\begin{array}{l} G\left(v'', \dfrac{\partial v''}{\partial X}\right) + G\left(v', \dfrac{\partial v''}{\partial X}\right) + \\ +G\left(v', \dfrac{\partial v'}{\partial X}\right) + G\left(v'', \dfrac{\partial v'}{\partial X}\right) \end{array}\right] \end{array}\right\} = 0. \qquad (22)$$

Учитывая (1.45) и сокращая (34, 35) на 2, получаем уравнения (4, 5), где

$$q = q'_o + q''_o, \qquad (23)$$

Taking into account (1.45) and cancelling (34, 35) by 2, we get the equations (4, 5), where

8.4. About sufficient conditions of extremum

Above we have proved for incompressible fluid , that the necessary conditions (18, 19) of the existence of extremum for the full action-2 functional are also sufficient conditions, if the integral

$$I = \int_0^T \left\{ \oint_V \Re_{22} dV \right\} dt \qquad (24)$$

has constant sign, where

$$\Re_{22} = -\mu b_v \Delta(b_v) - 2\rho v'' G(b_v). \qquad (25)$$

For compressible fluid the necessary conditions (18, 19) of the existence of extremum for the full action-2 functional are also sufficient conditions, if the integral (24) has constant sign , where, contrary to (25),

$$\Re_{22} = -\mu b_v \Delta(b_v) - \frac{\mu}{3} b_v \Omega(b_v) - 2\rho v'' G(b_v). \quad (26)$$

For closed systems with a flow of систем incompressible fluid we have shown above that the value (25) assumes the form

$$\Re_{22} = -\mu b_v \Delta(b_v). \quad (27)$$

Similarly, for closed systems with a flow of compressible fluid the value (26) assumes the form

$$\Re_{22} = -\mu b_v \Delta(b_v) - \frac{\mu}{3} b_v \Omega(b_v). \quad (28)$$

Let us consider now, similarly to (24), the integral

$$J = \int_0^T \left\{ \oint_V \Re'_{22} dV \right\} dt \quad (29)$$

where

$$\Re'_{22} = -\mu \cdot v \cdot \Delta(v) - \frac{\mu}{3} v \cdot \Omega(v). \quad (30)$$

(i.e. in this formula instead of the function b_v there is the function of speed). As the proof of the integral's constancy of sign must be valid for any function, it is enough to prove the constancy of sign of integral (29) with speeds. For this we must note that:

- the first term in (30) expresses the heat energy exuded by the fluid as the result of internal friction,
- the second tem in (30) is the heat energy exuded/absorbed by the fluid as the result of expansion\compression.

The first energy is positive regardless to the value of vector-function of speed with respect to the coordinates (A more exact proof of this fact for the first term is given in [4, 5]). The second term is equal to zero (as in our statement the temperature is not taken into account, i.e. assumed to be constant). Therefore, integral (24, 30) is positive on any iteration, which was required to show.

Thus, the Navier-Stokes equations for incompressible fluid have a global solution.

9. Discussion

Physical assumptions are often built on mathematical corollary facts. So it may be legitimate to build mathematical assumption on the base of physical facts. In this book there are several such places

2. The equations are derived on the base of the presented principle of general action extremum.
3. The main equation is divided into two independent equations based on a physical fact – the absence of energy flow through a closed system.
4. The exclusion of continuity conditions for closed systems is based in the physical fact – the continuity of fluid flow in a closed system
5. Usually in the problem formulation we indicate the boundaries of solution search and the boundary conditions – for speed, acceleration pressure on the boundaries These conditions usually are formed on the base of physical facts, for example – the fluid "adhesion" to the walls, the walls hardness, etc. In the presented method we <u>do not include the boundary conditions into the problem formulation</u> – they are found in the process of solution.

We may point also some possible directions of this approach development , for example

i. for problems of electro- and magneto-hydrodynamics
ii. for free surfaces dynamics (in changing boundaries for constant fluid volume).

The proof of global solution existence belongs to closed systems Practically, we must analyze the bounded and closed systems. Therefore some methods of formal transformation of non-closed systems into closed ones are also proposed, such as:

1. long pipe as the limit of ring pipe,
2. transformation of a limited pipe segment into closed system

Supplement. Certain formulas

Here we shall consider the proof of some formulas that were used in the main text. First of all we must remind that

$$\mathrm{div}(v) = \left[\frac{\partial v_x}{\partial x} + \frac{\partial v_y}{\partial y} + \frac{\partial v_z}{\partial z} \right], \qquad \text{(p1)}$$

$$\nabla p = \mathrm{grad}(p) = \left[\frac{\partial p}{\partial x}, \frac{\partial p}{\partial y}, \frac{\partial p}{\partial z} \right], \qquad \text{(p2)}$$

$$\Delta v_x = \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2}, \tag{p3}$$

$$\Delta v = \begin{bmatrix} \frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \\ \frac{\partial^2 v_y}{\partial x^2} + \frac{\partial^2 v_y}{\partial y^2} + \frac{\partial^2 v_y}{\partial z^2} \\ \frac{\partial^2 v_z}{\partial x^2} + \frac{\partial^2 v_z}{\partial y^2} + \frac{\partial^2 v_z}{\partial z^2} \end{bmatrix}, \tag{p4}$$

$$(v \cdot \nabla) = \left[v_x \frac{\partial}{\partial x} + v_y \frac{\partial}{\partial y} + v_z \frac{\partial}{\partial z} \right], \tag{p5}$$

$$(v \cdot \nabla) v = \begin{bmatrix} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{bmatrix}. \tag{p6}$$

From (2.5, 2.7a) it follows that

$$P_1 = \frac{\rho}{2} \frac{d}{dt} \left(v_x^2 + v_y^2 + v_z^2 \right), \tag{p7}$$

i.e.

$$P_1 = \rho v \frac{dv}{dt} \tag{p8}$$

Let us consider the function (2.7) or

$$\frac{P_5}{\rho} = \frac{1}{2} \begin{pmatrix} v_x \frac{d}{dx} \left(v_x^2 + v_y^2 + v_z^2 \right) \\ + v_y \frac{d}{dy} \left(v_x^2 + v_y^2 + v_z^2 \right) \\ + v_z \frac{d}{dz} \left(v_x^2 + v_y^2 + v_z^2 \right) \end{pmatrix} \tag{p9}$$

or

$$P_5 = \frac{\rho}{2} v \cdot \Delta\left(W^2\right). \tag{p9a}$$

Differentiating, we shall get:

$$\frac{P_5}{\rho} = \left\{ \begin{array}{l} v_x \left(v_x \dfrac{dv_x}{dx} + v_y \dfrac{dv_y}{dx} + v_z \dfrac{dv_z}{dx} \right) + \\ v_y \left(v_x \dfrac{dv_x}{dx} + v_y \dfrac{dv_y}{dx} + v_z \dfrac{dv_z}{dx} \right) + \\ v_z \left(v_x \dfrac{dv_x}{dx} + v_y \dfrac{dv_y}{dx} + v_z \dfrac{dv_z}{dx} \right) \end{array} \right\}. \tag{p10}$$

After rearranging the items, we get

$$\frac{P_5}{\rho} = \left\{ \begin{array}{l} v_x \left(v_x \dfrac{dv_x}{dx} + v_y \dfrac{dv_x}{dy} + v_z \dfrac{dv_x}{dz} \right) + \\ v_y \left(v_x \dfrac{dv_y}{dx} + v_y \dfrac{dv_y}{dy} + v_z \dfrac{dv_y}{dz} \right) + \\ v_z \left(v_x \dfrac{dv_z}{dx} + v_y \dfrac{dv_z}{dy} + v_z \dfrac{dv_z}{dz} \right) \end{array} \right\}. \tag{p11}$$

Let us denote:

$$\begin{gathered} g_x = \left(v_x \frac{dv_x}{dx} + v_y \frac{dv_x}{dy} + v_z \frac{dv_x}{dz} \right), \\ g_y = \left(v_x \frac{dv_y}{dx} + v_y \frac{dv_y}{dy} + v_z \frac{dv_y}{dz} \right), \\ g_z = \left(v_x \frac{dv_z}{dx} + v_y \frac{dv_z}{dy} + v_z \frac{dv_z}{dz} \right). \end{gathered} \tag{p12}$$

Let us consider the vector

$$G = \begin{Bmatrix} g_x \\ g_y \\ g_y \end{Bmatrix} \qquad \text{(p13)}$$

or

$$G = \begin{bmatrix} v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} \\ v_x \frac{\partial v_y}{\partial x} + v_y \frac{\partial v_y}{\partial y} + v_z \frac{\partial v_y}{\partial z} \\ v_x \frac{\partial v_z}{\partial x} + v_y \frac{\partial v_z}{\partial y} + v_z \frac{\partial v_z}{\partial z} \end{bmatrix}. \qquad \text{(p14)}$$

Note that

$$\frac{1}{2} G(v) = 2G(v/2) \qquad \text{(p14a)}$$

From (p11-p14) we get

$$P_5 / \rho = v \cdot G, \qquad \text{(p15)}$$

$$\frac{\partial P_5(v, G(v))}{\partial v} = \rho G(v), \qquad \text{(p16)}$$

Comparing (p6) and (p14), we find that

$$G(v) = (v \cdot \nabla) v. \qquad \text{(p18)}$$

Thus,

$$\frac{\partial P_5(v, G)}{\partial v} = \rho (v \cdot \nabla) v, \qquad \text{(p19)}$$

Comparing (p9a, p15, p18), we find that

$$\Delta\left(W^2\right) = 2 \cdot (v \cdot \nabla) \cdot v. \qquad \text{(p19a)}$$

As dynamic pressure is determined [2] by

$$P_d = \rho W^2 / 2, \qquad \text{(p19c)}$$

then from (p18, p19a) it follows that the gradient of dynamic pressure is

$$\Delta\left(P_d\right) = \rho \cdot G. \qquad \text{(p19d)}$$

Let us consider also

$$G(v+b) = G(v) + G(b) + G_1(v, b) + G_2(v, b), \qquad \text{(p20)}$$

where

$$G_1(v,b)=\begin{bmatrix} v_x \frac{\partial b_x}{\partial x}+v_y \frac{\partial b_x}{\partial y}+v_z \frac{\partial b_x}{\partial z} \\ v_x \frac{\partial b_y}{\partial x}+v_y \frac{\partial b_y}{\partial y}+v_z \frac{\partial b_y}{\partial z} \\ v_x \frac{\partial b_z}{\partial x}+v_y \frac{\partial b_z}{\partial y}+v_z \frac{\partial b_z}{\partial z} \end{bmatrix}, \qquad \text{(p20a)}$$

$$G_2(v,b)=\begin{bmatrix} b_x \frac{\partial v_x}{\partial x}+b_x \frac{\partial v_x}{\partial y}+b_x \frac{\partial v_x}{\partial z} \\ b_y \frac{\partial v_y}{\partial x}+b_y \frac{\partial v_y}{\partial y}+b_y \frac{\partial v_y}{\partial z} \\ b_z \frac{\partial v_z}{\partial x}+b_z \frac{\partial v_z}{\partial y}+b_z \frac{\partial v_z}{\partial z} \end{bmatrix}. \qquad \text{(p20в)}$$

If $b=a\cdot b_v$, then

$$G(v+a\cdot b_v)=G(v)+a^2G(b_v)+aG_1(v,b_v)+aG_2(v,b_v). \qquad \text{(p21)}$$

We have

$$\frac{\partial_o}{\partial v'}\left(v''\frac{dv'}{dt}\right)=-\frac{dv''}{dt},\ \frac{\partial_o}{\partial v''}\left(v''\frac{dv'}{dt}\right)=\frac{dv'}{dt},$$

$$\frac{\partial_o}{\partial v'}(v'\Delta v')=2\Delta v'',$$

$$\frac{\partial_o}{\partial v'}(v''G(v'))=-G\left(v',\frac{\partial v''}{\partial X}\right),\ \frac{\partial_o}{\partial v'}(v'G(v''))=G(v''),$$

$$\frac{\partial_o}{\partial v'}(v'\cdot\nabla(p''))=\nabla(p''),\ \frac{\partial_o}{\partial p''}(v'\cdot\nabla(p''))=-\mathrm{div}(v').$$

$$\frac{\partial_o}{\partial v'}\mathrm{div}(v'\cdot p'')=\nabla(p''),\frac{\partial_o}{\partial p''}\mathrm{div}(v'\cdot p'')=-\mathrm{div}(v')\text{-see. (p31).}$$

(p22)

The necessary conditions for extremum of functional from the functions with several independent variables – the Ostrogradsky equations [3] have for each of the functions the form

$$\frac{\partial_o f}{\partial v} = \frac{\partial f}{\partial v} - \sum_{a=x,y,z,t} \left[\frac{\partial}{\partial a} \left(\frac{\partial f}{\partial (dv/da)} \right) \right] = 0, \qquad \text{(p23)}$$

where f – the integration element, $v(x,y,z,t)$ – the variable function, a – independent variable.

$$\Omega(v) = \left[\frac{\partial(\mathrm{div}(v))}{\partial x}, \frac{\partial(\mathrm{div}(v))}{\partial y}, \frac{\partial(\mathrm{div}(v))}{\partial z} \right], \qquad \text{(p29)}$$

$$\Omega(v) = \begin{bmatrix} \dfrac{\partial^2 v_x}{\partial x^2} + \dfrac{\partial^2 v_y}{\partial x \partial y} + \dfrac{\partial^2 v_z}{\partial x \partial z} \\ \dfrac{\partial^2 v_x}{\partial x \partial y} + \dfrac{\partial^2 v_y}{\partial y^2} + \dfrac{\partial^2 v_z}{\partial y \partial z} \\ \dfrac{\partial^2 v_x}{\partial x \partial z} + \dfrac{\partial^2 v_y}{\partial y \partial z} + \dfrac{\partial^2 v_z}{\partial z^2} \end{bmatrix}, \qquad \text{(p30)}$$

If ρ, p are scalar fields, and v is a vector field, then

$$\mathrm{div}(\rho \cdot v) = v \cdot \mathrm{grad}(\rho) + \rho \cdot \mathrm{div}(v), \qquad \text{(p31)}$$

$$\mathrm{div}(\rho \cdot p \cdot v) = \rho \cdot v \cdot \mathrm{grad}(p) + p \cdot \mathrm{div}(\rho \cdot v), \qquad \text{(p32)}$$

i.e.

$$\mathrm{div}(\rho \cdot p \cdot v) = \rho \cdot v \cdot \mathrm{grad}(p) + p \cdot v \cdot \mathrm{grad}(\rho) + p \cdot \rho \cdot \mathrm{div}(v). \qquad \text{(p33)}$$

Consider $\mathrm{div}(\rho \cdot p' \cdot v'')$ and suppose that the extremum of a certain functional is determined or by varying the function p', or by varying the function v''. Then, differentiating the last expression by Ostrogradsky formula (p23), we shall find:

$$\frac{\partial_o}{\partial p'} [\mathrm{div}(\rho \cdot p' \cdot v'')] = 0 + v'' \cdot \mathrm{grad}(\rho) + \rho \cdot \mathrm{div}(v''),$$

$$\frac{\partial_o}{\partial v''} [\mathrm{div}(\rho \cdot p' \cdot v'')] = \rho \cdot \mathrm{grad}(p') + p' \cdot \mathrm{grad}(\rho) - p' \cdot \mathrm{grad}(\rho)$$

or

$$\frac{\partial_o}{\partial p'} [\mathrm{div}(\rho \cdot p' \cdot v'')] = \mathrm{div}(\rho \cdot v''), \qquad \text{(p34)}$$

$$\frac{\partial_o}{\partial v''} [\mathrm{div}(\rho \cdot p' \cdot v'')] = \rho \cdot \mathrm{grad}(p'). \qquad \text{(p35)}$$

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